

Irreducible Decomposition of Products of 10D Chiral Sigma Matrices ^{*}

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ABSTRACT

We review the enveloping algebra of the 10 dimensional chiral sigma matrices. To facilitate the computation of the product of several chiral sigma matrices we have developed a symbolic program. Using this program one can reduce the multiplication of the sigma matrices down to linear combinations of irreducible elements. We are able to quickly derive several identities that are not restricted to traces. A copy of the program written in the Mathematica language is provided for the community.

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I. SPINORS AND SIGMA MATRICES IN 10 D

Several problems in 10 D Physics require the ability to reduce products of the 10 D chiral sigma matrices into a linear combination of the irreducible matrices [1-4]. In general such calculations can quickly become intractable. For some problems, many approximation schemes are applied such as linearization and perturbation theory. However when one is discussing properties of algebras, such as the closing of the Jacobi identity or commutation relations, exact results are mandatory [5]. Computations such as computing the equations of motion in some supergravity theories, could literally take several weeks. These types of calculations could be reduced to a few hours work using today's powerful symbolic manipulators, such as Mathematica and Maple, in conjunction with high speed computers. In this note we provide the community with an example of such a program along with several identities that we have derived using this tool. We begin by outlining the formal construction of the enveloping algebra of the 10 dimensional chiral sigma matrices.

It is well known that in D-dimensional space-time Dirac spinor has $2^{\frac{D}{2}}$ complex components if D is even and in that case all the representations of the Dirac algebra are equivalent, while in odd dimensional space-time the number of components is $2^{\frac{D-1}{2}}$ with two inequivalent representations. This result is valid for any signature of the flat space-time metric. However, this kind of spinor is not, for any D, the smallest irreducible representation. When D is even we can always impose the Weyl condition which separates the spinor into its left and right handed components which transform independently under the Lorentz group. Each of these Weyl spinors thus has only half degrees of freedom of the original Dirac spinor. One more condition can be imposed to reduce further the number of independent components. This is so called Majorana condition which relates the charge-conjugate Dirac spinor to the original one. If after the charge conjugation the spinor is not changed, the number of independent components is reduced again by a factor of two. In that case the left and right Weyl spinors are related. The resulting spinor is necessarily real. This type of reduction can be done in $2 \pmod{8}$ dimensional space-time. It then follows that for $D = 10$ we can impose both conditions and have for the basic spinor a 16-dimensional real object. Consequently, instead of using the Dirac gamma matrices we use the Pauli matrices, of dimension 16×16 with real elements. We can then use small Greek index to denote 16 component left-handed Majorana-Weyl spinor and dotted index for a right-handed one:

$$(\psi^\alpha)^* = (\psi^\alpha), \quad (\chi^{\dot{\alpha}})^* = (\chi^{\dot{\alpha}}). \quad (1)$$

The indices can be raised and lowered by the use of the charge conjugation matrix which has mixed indices: $C_{\alpha\dot{\beta}}$ and $C^{\alpha\dot{\beta}}$,

$$\psi_{\dot{\beta}} = \psi^\alpha C_{\alpha\dot{\beta}}, \quad \chi_\alpha = \chi^{\dot{\beta}} C_{\alpha\dot{\beta}}, \quad (2)$$

$$C_{\alpha\dot{\gamma}} C^{\alpha\dot{\beta}} = \delta_{\dot{\gamma}}^{\dot{\beta}}, \quad C^{\alpha\dot{\beta}} C_{\gamma\dot{\beta}} = \delta_\gamma^\alpha. \quad (3)$$

As we can see, raising and lowering the indices also changes the handedness of a spinor. The Dirac algebra (of the Pauli matrices) is defined through the usual anticommutation relation:

$$(\sigma_{\underline{a}})_{\alpha\beta} (\sigma_{\underline{b}})^{\beta\gamma} + (\sigma_{\underline{b}})_{\alpha\beta} (\sigma_{\underline{a}})^{\beta\gamma} = -2\eta_{\underline{a}\underline{b}} \delta_\alpha^\gamma. \quad (4)$$

The sigma matrices can be regarded as bispinors. There are three types of them: purely left-handed, purely right-handed and mixed ones. The purely left-handed ones are

$$(\sigma^{\underline{a}})_{\alpha\beta}, \quad (\sigma^{\underline{abc}})_{\alpha\beta}, \quad (\sigma^{\underline{a_1 a_2 a_3 a_4 a_5}})_{\alpha\beta}, \quad (5)$$

where lower case Latin indices are vector indices denoting space-time directions. Purely right-handed bispinors have only dotted indices, but due to the existence of the charge conjugation matrix we have

$$(\sigma^{\underline{a}})^{\alpha\beta} = C^{\alpha\dot{\alpha}} C^{\beta\dot{\beta}} (\sigma^{\underline{a}})_{\dot{\alpha}\dot{\beta}}. \quad (6)$$

The mixed bispinors are

$$C_{\alpha\dot{\beta}}, \quad (\sigma^{\underline{ab}})_{\alpha\dot{\beta}}, \quad (\sigma^{\underline{abcd}})_{\alpha\dot{\beta}}. \quad (7)$$

They are, of course, related to

$$\delta_{\alpha}^{\beta}, \quad (\sigma^{\underline{ab}})_{\alpha}^{\beta}, \quad (\sigma^{\underline{abcd}})_{\alpha}^{\beta}. \quad (8)$$

The sigma matrices with more vector indices are defined through the following multiplication table:

$$\begin{aligned} (\sigma_{\underline{a}})_{\alpha\beta} (\sigma_{\underline{b}})^{\beta\gamma} &= -\eta_{\underline{ab}} \delta_{\alpha}^{\gamma} - (\sigma_{\underline{ab}})_{\alpha}^{\gamma}, \\ (\sigma_{\underline{a}})_{\alpha\beta} (\sigma_{\underline{bc}})^{\beta}_{\gamma} &= -\eta_{\underline{a}[\underline{b}} (\sigma_{\underline{c}]})_{\alpha\gamma} - (\sigma_{\underline{abc}})_{\alpha\gamma}, \\ (\sigma_{\underline{a}})_{\alpha\beta} (\sigma_{\underline{bcd}})^{\beta\gamma} &= -\frac{1}{2!} \eta_{\underline{a}[\underline{b}} (\sigma_{\underline{cd}]})_{\alpha}^{\gamma} - (\sigma_{\underline{abcd}})_{\alpha}^{\gamma}, \\ (\sigma_{\underline{a}})_{\alpha\beta} (\sigma_{\underline{bcde}})^{\beta}_{\gamma} &= \frac{1}{3!} \eta_{\underline{a}[\underline{b}} (\sigma_{\underline{cde]})_{\alpha\gamma} + (\sigma_{\underline{abcde}})_{\alpha\gamma}, \\ (\sigma_{\underline{a}})_{\alpha\beta} (\sigma_{\underline{bcdef}})^{\beta\gamma} &= -\frac{1}{4!} \eta_{\underline{a}[\underline{b}} (\sigma_{\underline{cdef]})_{\alpha}^{\gamma} - \frac{1}{4!} \epsilon_{\underline{abcdef}[4]} (\sigma^{[4]})_{\alpha}^{\gamma}. \end{aligned} \quad (9)$$

$$\begin{aligned} (\sigma_{\underline{a}})_{\alpha\beta} (\sigma_{\underline{b}})^{\beta\gamma} &= -\eta_{\underline{ab}} \delta_{\alpha}^{\gamma} - (\sigma_{\underline{ab}})_{\alpha}^{\gamma}, \\ (\sigma_{\underline{bc}})_{\beta}^{\alpha} (\sigma_{\underline{a}})^{\beta\gamma} &= +\eta_{\underline{a}[\underline{b}} (\sigma_{\underline{c]}})^{\alpha\gamma} - (\sigma_{\underline{abc}})^{\alpha\gamma}, \\ (\sigma_{\underline{bcd}})_{\alpha\beta} (\sigma_{\underline{a}})^{\beta\gamma} &= -\frac{1}{2!} \eta_{\underline{a}[\underline{b}} (\sigma_{\underline{cd]})_{\alpha}^{\gamma} + (\sigma_{\underline{abcd}})_{\alpha}^{\gamma}, \\ (\sigma_{\underline{bcde}})_{\beta}^{\alpha} (\sigma_{\underline{a}})^{\beta\gamma} &= -\frac{1}{3!} \eta_{\underline{a}[\underline{b}} (\sigma_{\underline{cde]})^{\alpha\gamma} + (\sigma_{\underline{abcde}})^{\alpha\gamma}, \\ (\sigma_{\underline{bcdef}})_{\alpha\beta} (\sigma_{\underline{a}})^{\beta\gamma} &= -\frac{1}{4!} \eta_{\underline{a}[\underline{b}} (\sigma_{\underline{cdef]})_{\alpha}^{\gamma} + \frac{1}{4!} \epsilon_{\underline{abcdef}[4]} (\sigma^{[4]})_{\alpha}^{\gamma}. \end{aligned} \quad (10)$$

All of the sigma matrices are totally antisymmetric in their vector indices. The sigma matrices with one or five vector indices are symmetric with respect to spinor indices, while the matrix with three vector indices is antisymmetric in spinor indices. The symmetrization and antisymmetrization is denoted in the following way:

$$\begin{aligned} A_{(\alpha} B_{\beta)} &= A_{\alpha} B_{\beta} + A_{\beta} B_{\alpha}, \\ A_{[\alpha} B_{\beta]} &= A_{\alpha} B_{\beta} - A_{\beta} B_{\alpha}. \end{aligned} \quad (11)$$

The sigma matrices with five vector indices satisfy the identities:

$$\begin{aligned} (\sigma_{[5]})_{\alpha\beta} &= \frac{1}{5!} \epsilon_{[5][\bar{5}]} (\sigma^{[\bar{5}]})_{\alpha\beta}, \\ (\sigma_{[5]})^{\alpha\beta} &= -\frac{1}{5!} \epsilon_{[5][\bar{5}]} (\sigma^{[\bar{5}]})^{\alpha\beta}. \end{aligned} \quad (12)$$

It is also useful to define matrices with more than five indices

$$\begin{aligned}
(\sigma_{[6]})_\alpha^\beta &= \frac{1}{4!} \epsilon_{[6][\bar{4}]} (\sigma^{[\bar{4}]})_\alpha^\beta, \\
(\sigma_{[7]})_{\alpha\beta} &= -\frac{1}{3!} \epsilon_{[7][\bar{3}]} (\sigma^{[\bar{3}]})_{\alpha\beta}, \\
(\sigma_{[7]})^{\alpha\beta} &= -\frac{1}{3!} \epsilon_{[7][\bar{3}]} (\sigma^{[\bar{3}]})^{\alpha\beta}, \\
(\sigma_{[8]})_\alpha^\beta &= -\frac{1}{2!} \epsilon_{[8][\bar{2}]} (\sigma^{[\bar{2}]})_\alpha^\beta, \\
(\sigma_{[9]})_{\alpha\beta} &= \epsilon_{[9]\underline{a}} (\sigma^{\underline{a}})_{\alpha\beta}, \\
(\sigma_{[9]})^{\alpha\beta} &= \epsilon_{[9]\underline{a}} (\sigma^{\underline{a}})^{\alpha\beta}.
\end{aligned} \tag{13}$$

The trace identities are:

$$\begin{aligned}
(\sigma_{\underline{a}})_{\alpha\beta} (\sigma^{\underline{c}})^{\alpha\beta} &= -16 \delta_{\underline{a}}^{\underline{c}}, \\
(\sigma_{\underline{ab}})_\alpha^\beta (\sigma^{\underline{cd}})_\beta^\alpha &= -16 \delta_{\underline{a}}^{\underline{c}} \delta_{\underline{b}}^{\underline{d}}, \\
(\sigma_{\underline{abc}})_{\alpha\beta} (\sigma^{\underline{def}})^{\alpha\beta} &= -16 \delta_{\underline{a}}^{\underline{d}} \delta_{\underline{b}}^{\underline{e}} \delta_{\underline{c}}^{\underline{f}}, \\
(\sigma_{\underline{abcd}})_\alpha^\beta (\sigma^{\underline{efgh}})_\beta^\alpha &= 16 \delta_{\underline{a}}^{\underline{e}} \delta_{\underline{b}}^{\underline{f}} \delta_{\underline{c}}^{\underline{g}} \delta_{\underline{d}}^{\underline{h}}, \\
(\sigma_{\underline{abcde}})_{\alpha\beta} (\sigma^{\underline{fghij}})^{\alpha\beta} &= -16 [\delta_{\underline{a}}^{\underline{f}} \delta_{\underline{b}}^{\underline{g}} \delta_{\underline{c}}^{\underline{h}} \delta_{\underline{d}}^{\underline{i}} \delta_{\underline{e}}^{\underline{j}} + \epsilon_{\underline{abcde}} \underline{fghij}].
\end{aligned} \tag{14}$$

In terms of ordinary matrices, our notation can be regarded as follows. We can introduce a bar-notation to distinguish the altitude of the indices on the bi-spinors in Eq(5). In otherwords we define the matrices

$$\sigma_{\underline{a}} \equiv (\sigma_{\underline{a}})_{\alpha\beta} \quad , \quad \bar{\sigma}_{\underline{a}} \equiv (\sigma_{\underline{a}})^{\alpha\beta} \quad , \tag{15}$$

and similarly for the third and fifth rank space-time tensors. We further use a Northwest-Southeast convention with respect to the contraction of hidden spinor indices. This is simplest to understand via the following examples

$$\sigma_{\underline{a}} \bar{\sigma}_{\underline{b}} \equiv -(\sigma_{\underline{a}})_{\alpha\beta} (\sigma_{\underline{b}})^{\beta\gamma} \quad , \quad \bar{\sigma}_{\underline{b}} \sigma_{\underline{a}} \equiv (\sigma_{\underline{b}})^{\gamma\beta} (\sigma_{\underline{a}})_{\beta\alpha} \quad . \tag{16}$$

So when the contraction is up-to-down there is a plus sign when writing out the explicit indices, but when the contraction is down-to-up there is a minus sign.

The sigma matrices with more vector indices follow from the multiplication table:

$$\begin{aligned}
\sigma_{\underline{a}} \bar{\sigma}_{\underline{b}} &= \eta_{\underline{ab}} + \sigma_{\underline{ab}} \quad , \\
\bar{\sigma}_{\underline{a}} \sigma_{\underline{b}} &= -\eta_{\underline{ab}} + \sigma_{\underline{ab}} \quad , \\
\sigma_{\underline{a}} \sigma_{\underline{bc}} &= \eta_{\underline{a}[\underline{b}\sigma_{\underline{c}]} + \sigma_{\underline{abc}} \quad , \\
\sigma_{\underline{bc}} \sigma_{\underline{a}} &= -\eta_{\underline{a}[\underline{b}\sigma_{\underline{c}]} + \sigma_{\underline{abc}} \quad , \\
\sigma_{\underline{a}} \bar{\sigma}_{\underline{bcd}} &= \frac{1}{2!} \eta_{\underline{a}[\underline{b}\sigma_{\underline{cd}]} + \sigma_{\underline{abcd}} \quad , \\
\bar{\sigma}_{\underline{bcd}} \sigma_{\underline{a}} &= \frac{1}{2!} \eta_{\underline{a}[\underline{b}\sigma_{\underline{cd}]} + \sigma_{\underline{abcd}} \quad , \\
\sigma_{\underline{a}} \sigma_{\underline{bcde}} &= -\frac{1}{3!} \eta_{\underline{a}[\underline{b}\sigma_{\underline{cde}]} - \sigma_{\underline{abcde}} \quad ,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\sigma_{\underline{bcde}}\sigma_{\underline{a}} &= -\frac{1}{3!}\eta_{\underline{a}[\underline{b}\sigma_{\underline{cde}}]} + \sigma_{\underline{abcde}} \quad , \\
\sigma_{\underline{a}}\bar{\sigma}_{\underline{bcdef}} &= \frac{1}{4!}\eta_{\underline{a}[\underline{b}\sigma_{\underline{cdef}}]} + \frac{1}{4!}\epsilon_{\underline{abcdef}[4]}\sigma^{[4]} \quad , \\
\bar{\sigma}_{\underline{bcdef}}\sigma_{\underline{a}} &= -\frac{1}{4!}\eta_{\underline{a}[\underline{b}\sigma_{\underline{cdef}}]} - \frac{1}{4!}\epsilon_{\underline{abcdef}[4]}\sigma^{[4]} \quad . \\
\\
\bar{\sigma}_{\underline{a}}\sigma_{\underline{bc}} &= \eta_{\underline{a}[\underline{b}\bar{\sigma}_{\underline{c}}]} + \bar{\sigma}_{\underline{abc}} \quad , \\
\sigma_{\underline{bc}}\bar{\sigma}_{\underline{a}} &= -\eta_{\underline{a}[\underline{b}\bar{\sigma}_{\underline{c}}]} + \bar{\sigma}_{\underline{abc}} \quad , \\
\bar{\sigma}_{\underline{a}}\sigma_{\underline{bcd}} &= \frac{1}{2!}\eta_{\underline{a}[\underline{b}\sigma_{\underline{cd}}]} - \sigma_{\underline{abcd}} \quad , \\
\sigma_{\underline{bcd}}\bar{\sigma}_{\underline{a}} &= \frac{1}{2!}\eta_{\underline{a}[\underline{b}\sigma_{\underline{cd}}]} - \sigma_{\underline{abcd}} \quad , \\
\bar{\sigma}_{\underline{a}}\sigma_{\underline{bcde}} &= \frac{1}{3!}\eta_{\underline{a}[\underline{b}\bar{\sigma}_{\underline{cde}}]} + \bar{\sigma}_{\underline{abcde}} \quad , \\
\sigma_{\underline{bcde}}\bar{\sigma}_{\underline{a}} &= \frac{1}{3!}\eta_{\underline{a}[\underline{b}\bar{\sigma}_{\underline{cde}}]} - \bar{\sigma}_{\underline{abcde}} \quad , \\
\bar{\sigma}_{\underline{a}}\sigma_{\underline{bcdef}} &= -\frac{1}{4!}\eta_{\underline{a}[\underline{b}\sigma_{\underline{cdef}}]} + \frac{1}{4!}\epsilon_{\underline{abcdef}[4]}\sigma^{[4]} \quad , \\
\sigma_{\underline{bcdef}}\bar{\sigma}_{\underline{a}} &= \frac{1}{4!}\eta_{\underline{a}[\underline{b}\sigma_{\underline{cdef}}]} - \frac{1}{4!}\epsilon_{\underline{abcdef}[4]}\sigma^{[4]} \quad .
\end{aligned} \tag{18}$$

This establishes the enveloping algebra of the 10 D chiral matrices. It is easy to see that one can use these rules to reduce any product of such matrices down to a linear combination of irreducible elements. However the presence of the two copies of five forms down to one forms renders many of the calculations intractable. In what follows we will introduce a program using the Mathematica language that can facilitate such computations. We will first give an example worksheet that shows how to use the program then present results that have exploited the routine. In the last section we present the program verbatim. We will call the program SigmaVector10D.m.

II. INSTRUCTIONS FOR USING SIGMAVECTOR10D.M

Here is an example run of the Mathematica package SigmaVector10D.m. First one must copy the program over to a Mathematica notebook and save the file as **SigmaVector10D.m**. Then open another notebook and bring in this file by typing, "<< **SigmaVector.m**". Below is a transcript of a typical run.

```
In[1]:= <<SigmaVector10D.m
Out[1]=
```

This is SigmaVector program. It reduces the product of sigma matrices. Spinor indices of the chiral sigma matrices analyzed here are suppressed. Sigma matrices with spinor indices down are denoted by S, as well as those matrices which have one spinor index down and other up. If both spinor indices are up, the letter SB is used. The program handles all contraction possibilities of spinor indices, except when two sigma matrices, both with even number of vector indices are multiplied in such a way that the lower index of the first sigma matrix is summed with the upper index of the second matrix. In this case the order of sigma matrices should be reversed. Here is an example: $(\sigma^{ef})_{\alpha}^{\beta} * (\sigma_{abc})^{\alpha\gamma} * (\sigma_{ef})_{\gamma}^{\delta}$ should be represented in the symbolic form as: **ExpandAll[ExpandAll[S[e, f] ** SB[a, b, c]] ** S[e, f] /. Rule1]**, where Rule1 is a rule which

transforms sigma matrices with 6 or more vector indices in appropriate sigma matrices with 4 or less vector indices and contracts corresponding epsilon symbols.

In[2]:= Here is the first example:

In[3]:= **Expand**[S[a,b] ** S[a,b,c]]

Out[3]= -2 S[c] + 3 Ten S[c] - Ten² S[c]

In[4]:= The output tells you something about the Kronecker delta symbols. In most cases you are not interested in that, so you type:

In[5]:= **Ten = 10**

Out[5]= 10

In[6]:= The previous example now becomes:

In[7]:= **Expand**[S[a,b] ** S[a,b,c]]

Out[7]= -72 S[c]

In[8]:= **ExpandAll**[

ExpandAll[S[e,f] ** S[a]] ** S[e,f]]

Out[8]= 54 S[a]

In[9]:=

In[10]:= Note that we have grouped the products of sigma matrices. This reduces the run time by a factor of three! When you have more than 10 vector indices, this is very important. In this simple example we could have used the following command:

In[11]:= **S[e,f] ** S[a] ** S[e,f]**

Out[11]= 54 S[a]

In[12]:= Here comes a nontrivial example:

In[13]:= **ExpandAll**[

ExpandAll[SB[e,f,g] ** S[a,b,c]] **

SB[e,f,g]

Out[13]= 258 SB[a,b,c] + $\frac{1}{24}$ Epsilon[a,b,c,e,

f,g,\$1[1],\$2[1],\$3[1],\$4[1]]

SB[e,f,g,\$1[1],\$2[1],\$3[1],\$4[1]]

In[14]:= We have obtained a sigma matrix with 7 vector indices!

The introduction of such matrices is important - the run time is much shorter. We now want to transform this 7-index matrix into a product of epsilon symbol and a matrix with 3 indices by the use of "Rule1". Two epsilon symbols will be automatically contracted.

In[15]:= **%//.Rule1**

Out[15]= 258 SB[a,b,c]—

$$\frac{1}{144} (-5040 \text{Id}[a, \$3[6]] \text{Id}[b, \$2[6]] \\
\text{Id}[c, \$1[6]] + 5040 \text{Id}[a, \$2[6]] \\
\text{Id}[b, \$3[6]] \text{Id}[c, \$1[6]] + \\
5040 \text{Id}[a, \$3[6]] \\
\text{Id}[b, \$1[6]] \text{Id}[c, \$2[6]] - \\
5040 \text{Id}[a, \$1[6]] \text{Id}[b, \$3[6]] \\
\text{Id}[c, \$2[6]] - 5040 \text{Id}[a, \$2[6]] \\
\text{Id}[b, \$1[6]] \text{Id}[c, \$3[6]] + \\
5040 \text{Id}[a, \$1[6]] \\
\text{Id}[b, \$2[6]] \text{Id}[c, \$3[6]]) \\
\text{SB}[\$1[6], \$2[6], \$3[6]]$$

In[16]:= In the previous input command line the % symbol stands for "the last output". Similarly, %% stands for the output before the last one. Better way is to use the output name directly, like in Out[17]//.Rule1. The command //.Rule1 means: apply Rule1 as many times as it is applicable. If you want to apply it only once, the appropriate command is /.Rule1. What we have obtained in the last output is the correct answer in an unwanted form. To get the final answer we type:

In[17]:= **ExpandAll**[%]

Out[17]= 48 SB[a,b,c]

In[18]:= When we put all this together, we see that the optimal command should have been:

In[19]:= **ExpandAll**[
ExpandAll[SB[e,f,g] ** S[a,b,c]] **
SB[e,f,g]//.
Rule1]

Out[19]= 48 SB[a,b,c]

In[20]:= It is always better to use ExpandAll then Expand. Note also that Rule1 has been used only once and at the end of the calculation! If you use it more than once in the same command line you will get wrong answer! Sometimes, you want to use only a portion of the intermediate result and multiply it with another sigma matrix - to check the result, for example.

In[21]:= **ExpandAll**[SB[e,f,g] ** S[a,b,c]]

```

Out[21]= -(Eta[a,g] Eta[b,f] Eta[c,e])+
          Eta[a,f] Eta[b,g] Eta[c,e]+
          Eta[a,g] Eta[b,e] Eta[c,f]-
          Eta[a,e] Eta[b,g] Eta[c,f]-
          Eta[a,f] Eta[b,e] Eta[c,g]+
          Eta[a,e] Eta[b,f] Eta[c,g]-
          Eta[b,g] Eta[c,f] S[a,e]+
Eta[b,f] Eta[c,g] S[a,e]+
Eta[b,g] Eta[c,e] S[a,f]-
Eta[b,e] Eta[c,g] S[a,f]-
Eta[b,f] Eta[c,e] S[a,g]+
Eta[b,e] Eta[c,f] S[a,g]+
Eta[a,g] Eta[c,f] S[b,e]-
Eta[a,f] Eta[c,g] S[b,e]-
Eta[a,g] Eta[c,e] S[b,f]+
Eta[a,e] Eta[c,g] S[b,f]+
Eta[a,f] Eta[c,e] S[b,g]-
Eta[a,e] Eta[c,f] S[b,g]-
Eta[a,g] Eta[b,f] S[c,e]+
Eta[a,f] Eta[b,g] S[c,e]+
Eta[a,g] Eta[b,e] S[c,f]-
Eta[a,e] Eta[b,g] S[c,f]-
Eta[a,f] Eta[b,e] S[c,g]+
Eta[a,e] Eta[b,f] S[c,g]-
Eta[c,g] S[a,b,e,f]+
Eta[c,f] S[a,b,e,g]-
Eta[c,e] S[a,b,f,g]+
Eta[b,g] S[a,c,e,f]-
Eta[b,f] S[a,c,e,g]+
Eta[b,e] S[a,c,f,g]-
Eta[a,g] S[b,c,e,f]+
Eta[a,f] S[b,c,e,g]-
Eta[a,e] S[b,c,f,g]-
S[a,b,c,e,f,g]

```

In[22]:= Eta symbol is the same as Id - depending whether both vector indices are down or up or not. Suppose now that you want to extract the last term S[a,b,c,e,f,g] and to multiply it with SB[e,f,g]. You first click the mouse on the last output and press "Command-x", i.e. the Command key followed by x. This cuts the output. Put it back by pressing "Command-v". This will paste it back. Move the cursor beneath the last cell and press again

"Command-v".


```

Out[22]= -(Eta[a,g] Eta[b,f] Eta[c,e])+
          Eta[a,f] Eta[b,g] Eta[c,e]+
          Eta[a,g] Eta[b,e] Eta[c,f]-
          Eta[a,e] Eta[b,g] Eta[c,f]-
          Eta[a,f] Eta[b,e] Eta[c,g]+
          Eta[a,e] Eta[b,f] Eta[c,g]-
          Eta[b,g] Eta[c,f] S[a,e]+
          Eta[b,f] Eta[c,g] S[a,e]+
          Eta[b,g] Eta[c,e] S[a,f]-
          Eta[b,e] Eta[c,g] S[a,f]-
          Eta[b,f] Eta[c,e] S[a,g]+
          Eta[b,e] Eta[c,f] S[a,g]+
          Eta[a,g] Eta[c,f] S[b,e]-
          Eta[a,f] Eta[c,g] S[b,e]-
          Eta[a,g] Eta[c,e] S[b,f]+
          Eta[a,e] Eta[c,g] S[b,f]+
          Eta[a,f] Eta[c,e] S[b,g]-
          Eta[a,e] Eta[c,f] S[b,g]-
          Eta[a,g] Eta[b,f] S[c,e]+
          Eta[a,f] Eta[b,g] S[c,e]+
          Eta[a,g] Eta[b,e] S[c,f]-
          Eta[a,e] Eta[b,g] S[c,f]-
          Eta[a,f] Eta[b,e] S[c,g]+
          Eta[a,e] Eta[b,f] S[c,g]-
          Eta[c,g] S[a,b,e,f]+
          Eta[c,f] S[a,b,e,g]-
          Eta[c,e] S[a,b,f,g]+
          Eta[b,g] S[a,c,e,f]-
          Eta[b,f] S[a,c,e,g]+
          Eta[b,e] S[a,c,f,g]-
          Eta[a,g] S[b,c,e,f]+
          Eta[a,f] S[b,c,e,g]-
          Eta[a,e] S[b,c,f,g]-
          S[a,b,c,e,f,g]

```

In[23]:= Now you want this output to be turned into an input -

bold face. Click again on it and press "Command 9". This will turn it into a bold-faced form:

```

In[24]:= -(Eta[a, g] Eta[b, f] Eta[c, e]) +
          Eta[a, f] Eta[b, g] Eta[c, e] +
          Eta[a, g] Eta[b, e] Eta[c, f] -
          Eta[a, e] Eta[b, g] Eta[c, f] -
          Eta[a, f] Eta[b, e] Eta[c, g] +
          Eta[a, e] Eta[b, f] Eta[c, g] -
          Eta[b, g] Eta[c, f] S[a, e] +
          Eta[b, f] Eta[c, g] S[a, e] +
          Eta[b, g] Eta[c, e] S[a, f] -
          Eta[b, e] Eta[c, g] S[a, f] -
          Eta[b, f] Eta[c, e] S[a, g] +
          Eta[b, e] Eta[c, f] S[a, g] +
          Eta[a, g] Eta[c, f] S[b, e] -
          Eta[a, f] Eta[c, g] S[b, e] -
          Eta[a, g] Eta[c, e] S[b, f] +
          Eta[a, e] Eta[c, g] S[b, f] +
          Eta[a, f] Eta[c, e] S[b, g] -
          Eta[a, e] Eta[c, f] S[b, g] -
          Eta[a, g] Eta[b, f] S[c, e] +
          Eta[a, f] Eta[b, g] S[c, e] +
          Eta[a, g] Eta[b, e] S[c, f] -
          Eta[a, e] Eta[b, g] S[c, f] -
          Eta[a, f] Eta[b, e] S[c, g] +
          Eta[a, e] Eta[b, f] S[c, g] -
          Eta[c, g] S[a, b, e, f] +
          Eta[c, f] S[a, b, e, g] -
          Eta[c, e] S[a, b, f, g] +
          Eta[b, g] S[a, c, e, f] -
          Eta[b, f] S[a, c, e, g] +
          Eta[b, e] S[a, c, f, g] -
          Eta[a, g] S[b, c, e, f] +
          Eta[a, f] S[b, c, e, g] -
          Eta[a, e] S[b, c, f, g] -
          S[a, b, c, e, f, g]

```

In[25]:= You can now darken the part which you want to throw out:
Click the mouse at the beginning of the last input and
drag it until the last term. Cut the darkened piece
by "Command x". What is left is S[a,b,c,e,f,g].

In[26]:= Below are two examples that correct Eqs(40) and (43) in the
appendix of [6]:

```

In[27]:= ExpandAll[SB[f, a, b] * S[f, c, d]]

```

```

Out[27]= -8 Eta[a, d] Eta[b, c]+
          8 Eta[a, c] Eta[b, d] - 7 Eta[b, d] S[a, c]+
          7 Eta[b, c] S[a, d] + 7 Eta[a, d] S[b, c]-
          7 Eta[a, c] S[b, d] - 6 S[a, b, c, d]
In[28]:= ExpandAll[SB[a, b, c, d, e] ** S[a, f, g]]
Out[28]= 6 Eta[d, g] Eta[e, f] S[b, c]-
          6 Eta[d, f] Eta[e, g] S[b, c]-
          6 Eta[c, g] Eta[e, f] S[b, d]+
          6 Eta[c, f] Eta[e, g] S[b, d]+
          6 Eta[c, g] Eta[d, f] S[b, e]-
          6 Eta[c, f] Eta[d, g] S[b, e]+
          6 Eta[b, g] Eta[e, f] S[c, d]-
          6 Eta[b, f] Eta[e, g] S[c, d]-
          6 Eta[b, g] Eta[d, f] S[c, e]+
          6 Eta[b, f] Eta[d, g] S[c, e]+
          6 Eta[b, g] Eta[c, f] S[d, e]-
          6 Eta[b, f] Eta[c, g] S[d, e]+
          5 Eta[e, g] S[b, c, d, f]-
          5 Eta[e, f] S[b, c, d, g]-
          5 Eta[d, g] S[b, c, e, f]+
          5 Eta[d, f] S[b, c, e, g]+
          5 Eta[c, g] S[b, d, e, f]-
          5 Eta[c, f] S[b, d, e, g]-
          5 Eta[b, g] S[c, d, e, f]+
          5 Eta[b, f] S[c, d, e, g]+
          4 S[b, c, d, e, f, g]

```

III. IDENTITIES AND CALCULATED RESULTS

Before using the program to find new identities we use the definitions from the first section to construct the following Fiertz identities. Note that our program will not derive these identities since they are not products of the elements via matrix multiplication.

$$(\sigma^{\underline{a}})_{(\alpha\beta}(\sigma_{\underline{a}})_{\gamma)\delta} = 0, \quad (19)$$

$$(\sigma^{\underline{a}})_{\alpha(\beta}(\sigma_{\underline{a}})_{\gamma)\delta} = -(\sigma^{\underline{a}})_{\beta\gamma}(\sigma_{\underline{a}})_{\alpha\delta}, \quad (20)$$

$$(\sigma^{\underline{a}})^{\alpha(\beta}(\sigma_{\underline{a}})^{\gamma)\delta} = -(\sigma^{\underline{a}})^{\beta\gamma}(\sigma_{\underline{a}})^{\alpha\delta}, \quad (21)$$

$$(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})_{\gamma\delta} = 0, \quad (22)$$

$$(\sigma^{\underline{abc}})_{\alpha\beta}(\sigma_{\underline{abc}})^{\gamma\delta} = -8 \cdot 3! \delta_{[\alpha}^{\gamma} \delta_{\beta]}^{\delta}, \quad (23)$$

$$(\sigma^{\underline{abc}})_{\alpha\beta}(\sigma_{\underline{abc}})_{\gamma\delta} = -2 \cdot 3! (\sigma^{\underline{a}})_{\alpha[\gamma}(\sigma_{\underline{a}})_{\delta]\beta}, \quad (24)$$

$$(\sigma^{\underline{abc}})^{\alpha\beta}(\sigma_{\underline{abc}})^{\gamma\delta} = -2 \cdot 3! (\sigma^{\underline{a}})^{\alpha[\gamma}(\sigma_{\underline{a}})^{\delta]\beta}, \quad (25)$$

$$(\sigma^{[4]})_{\alpha}^{\gamma}(\sigma_{[4]})_{\beta}^{\delta} = 4! \{-2 \delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} + 12 \delta_{\beta}^{\gamma} \delta_{\alpha}^{\delta} - 2 (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}\}, \quad (26)$$

$$(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\gamma\delta} = 5! \{-16 \delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} - 2 (\sigma^{\underline{a}})_{\alpha\beta} (\sigma_{\underline{a}})^{\gamma\delta}\}, \quad (27)$$

$$(\sigma^{\underline{ab}})_{\alpha}^{\gamma} (\sigma_{\underline{ab}})_{\beta}^{\delta} = 2 \{-\delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} - 4 \delta_{\beta}^{\gamma} \delta_{\alpha}^{\delta} - 2 (\sigma^{\underline{a}})_{\alpha\beta} (\sigma_{\underline{a}})^{\gamma\delta}\}, \quad (28)$$

$$(\sigma^{\underline{ab}})_{[\alpha}^{\gamma} (\sigma_{\underline{ab}})_{\beta]}^{\delta} = 6 \delta_{[\alpha}^{\gamma} \delta_{\beta]}^{\delta}, \quad (29)$$

$$(\sigma^{\underline{ab}})_{(\alpha}^{\gamma} (\sigma_{\underline{ab}})_{\beta)}^{\delta} = -10 \delta_{(\alpha}^{\gamma} \delta_{\beta)}^{\delta} - 8 (\sigma^{\underline{a}})_{\alpha\beta} (\sigma_{\underline{a}})^{\gamma\delta}, \quad (30)$$

$$[2 \delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} + (\sigma^{\underline{b}})_{(\alpha\beta} (\sigma_{\underline{b}})^{\gamma\delta}] (\sigma_{\underline{a}})_{\epsilon}^{\delta} = 0, \quad (31)$$

$$\{(\sigma_{\underline{a}})_{(\alpha|\gamma} (\sigma^{\underline{efg}})^{\gamma\delta} + \frac{1}{2} (\sigma^{\underline{efg}})_{(\alpha|\gamma} (\sigma_{\underline{a}})^{\gamma\delta}) (\sigma_{\underline{bcd}})_{\beta)}^{\delta} = 48 (\sigma_{\underline{a}})_{\alpha\beta} \delta_{\underline{b}}^{\epsilon} \delta_{\underline{c}}^f \delta_{\underline{d}}^g, \quad (32)$$

$$(\sigma_{\underline{abc}})_{(\alpha(\beta} (\sigma_{\underline{c}})^{\gamma)}_{\delta)} = -2 (\sigma_{\underline{a}})_{\alpha\delta} (\sigma_{\underline{b}})_{\beta\gamma}, \quad (33)$$

$$(\sigma^{\underline{a}})_{(\alpha\beta} (\sigma_{\underline{ab}})_{\gamma)}^{\delta} = -(\sigma_{\underline{b}})_{(\alpha\beta} \delta_{\gamma)}^{\delta}, \quad (34)$$

$$(\sigma^{\underline{a}})_{(\alpha\beta} (\sigma_{\underline{ab}})_{\delta)}^{\gamma} = (\sigma_{\underline{b}})_{(\alpha\beta} \delta_{\delta)}^{\gamma}, \quad (35)$$

$$(\sigma^{\underline{c}})_{(\alpha\beta} (\sigma_{\underline{ab}})_{\gamma)}^{\delta} (\sigma_{\underline{c}})_{\delta\epsilon} = -2 (\sigma_{\underline{a}})_{(\alpha\beta} (\sigma_{\underline{b}})_{\gamma)}^{\epsilon}. \quad (36)$$

There is also an interesting identity for which it is convenient to first introduce two arbitrary three-forms in order to write the identity in a simple manner,

$$\begin{aligned} (\sigma^{[3]\underline{kl}})_{(\alpha\beta|} (\sigma_{[3']\underline{kl}})_{|\gamma\delta)} A_{[3]} B_{[3']} &= (\sigma^{\underline{a}})_{(\alpha\beta|} (\sigma_{\underline{b}})_{|\gamma\delta)} A_{\underline{acd}} B_{\underline{b}}^{\underline{cd}} \\ &\quad - 6 (\sigma^{\underline{a}})_{(\alpha\beta|} (\sigma_{\underline{b}})_{|\gamma\delta)} (A_{\underline{abc}} B_{\underline{def}} + B_{\underline{abc}} A_{\underline{def}}), \end{aligned} \quad (37)$$

valid for the arbitrary $A_{[3]} \equiv A_{\underline{abc}}$ and $B_{[3']} \equiv B_{\underline{def}}$.

Finally, the following identities for manipulating the σ -matrices are consequence of our definitions and are a direct test of the reduction program, SigmaVector10D.m.

$$(\sigma_{\underline{bcd}})_{\alpha\beta} (\sigma^{\underline{abc}})^{\beta\gamma} = 72 \delta_{\underline{a}}^{\gamma} \delta_{\alpha}^{\gamma} + 56 (\sigma_{\underline{a}}^{\gamma})_{\alpha}^{\gamma}, \quad (38)$$

$$(\sigma^{\underline{abcd}})_{\alpha}^{\beta} (\sigma_{\underline{bcd}})_{\beta\gamma} = -7 \cdot 8 \cdot 9 (\sigma^{\underline{a}})_{\alpha\gamma}, \quad (39)$$

$$(\sigma^{\underline{abc}})^{\alpha\beta} (\sigma_{\underline{c}})_{\beta\gamma} = 8 (\sigma^{\underline{ab}})_{\gamma}^{\alpha}, \quad (40)$$

$$(\sigma^{\underline{abc}})_{\alpha\beta} (\sigma_{\underline{c}})^{\beta\gamma} = -8 (\sigma^{\underline{ab}})_{\alpha}^{\gamma}, \quad (41)$$

$$(\sigma_{\underline{a}})_{\alpha\beta} (\sigma^{\underline{ab}})_{\gamma}^{\beta} = -9 (\sigma^{\underline{b}})_{\alpha\gamma}, \quad (42)$$

$$(\sigma_{\underline{a}})^{\alpha\beta} (\sigma^{\underline{ab}})_{\beta}^{\gamma} = 9 (\sigma^{\underline{b}})^{\alpha\gamma}, \quad (43)$$

$$(\sigma^{\underline{abcd}})_{\alpha}^{\beta} (\sigma_{\underline{a}})_{\beta\gamma} = -7 (\sigma^{\underline{bcd}})_{\alpha\gamma}, \quad (44)$$

$$(\sigma^{\underline{abcd}})_{\beta}^{\alpha} (\sigma_{\underline{a}})^{\beta\gamma} = -7 (\sigma^{\underline{bcd}})^{\alpha\gamma}, \quad (45)$$

$$(\sigma_{\underline{ab}})_{\alpha}^{\beta} (\sigma^{\underline{cd}})_{\beta}^{\gamma} = -\delta_{[\underline{a}}^{\underline{c}} \delta_{\underline{b}}^{\underline{d}]} \delta_{\alpha}^{\gamma} - \delta_{[\underline{a}}^{\underline{c}} (\sigma_{\underline{b}})^{\underline{d}]}_{\alpha}^{\gamma} + (\sigma_{\underline{ab}})^{\underline{cd}}_{\alpha}^{\gamma}, \quad (46)$$

$$(\sigma_{\underline{abc}})_{\alpha\beta} (\sigma^{\underline{def}})^{\beta\gamma} = \delta_{[\underline{a}}^{\underline{d}} \delta_{\underline{b}}^{\underline{e}} \delta_{\underline{c}}^{\underline{f}]} \delta_{\alpha}^{\gamma} + \frac{1}{2} \delta_{[\underline{a}}^{\underline{d}} \delta_{\underline{b}}^{\underline{e}} (\sigma_{\underline{c}})^{\underline{f}]}_{\alpha}^{\gamma} - \frac{1}{4} \delta_{[\underline{a}}^{\underline{d}} (\sigma_{\underline{bc}})^{\underline{ef}]}_{\alpha}^{\gamma} - (\sigma_{\underline{abc}})^{\underline{def}}_{\alpha}^{\gamma}, \quad (47)$$

$$(\sigma^{\underline{ab}})_{\alpha}^{\beta} (\sigma_{\underline{abc}})_{\beta\gamma} = -72 (\sigma_{\underline{c}})_{\alpha\gamma}, \quad (48)$$

$$(\sigma_{\underline{abc}})^{\alpha\beta} (\sigma^{\underline{ab}})_{\beta}^{\gamma} = -72 (\sigma_{\underline{c}})^{\alpha\gamma}, \quad (49)$$

$$(\sigma^{\underline{abcde}})^{\alpha\beta} (\sigma_{\underline{ab}})_{\beta}^{\gamma} = -6 \cdot 7 (\sigma^{\underline{cde}})^{\alpha\gamma}, \quad (50)$$

$$(\sigma^{\underline{abcde}})_{\alpha\beta} (\sigma_{\underline{ab}})_{\gamma}^{\beta} = 6 \cdot 7 (\sigma^{\underline{cde}})_{\alpha\gamma}, \quad (51)$$

$$(\sigma^{\underline{ab}})_{\alpha}^{\beta} (\sigma_{\underline{ab}})^{\underline{cd}}_{\beta}^{\gamma} = -7 \cdot 8 (\sigma^{\underline{cd}})_{\alpha}^{\gamma}, \quad (52)$$

$$(\sigma_{\underline{ab}})_{\beta}^{\alpha} (\sigma_{\underline{ab}})^{\underline{cd}}_{\gamma}^{\beta} = -7 \cdot 8 (\sigma^{\underline{cd}})_{\gamma}^{\alpha}, \quad (53)$$

$$(\sigma^{\underline{abcde}})^{\alpha\beta} (\sigma_{\underline{abc}})_{\beta\gamma} = -6 \cdot 7 \cdot 8 (\sigma^{\underline{de}})_{\gamma}^{\alpha}, \quad (54)$$

$$(\sigma^{\underline{abcde}})_{\alpha\beta} (\sigma_{\underline{abc}})_{\gamma}^{\beta} = 6 \cdot 7 \cdot 8 \cdot 9 (\sigma^{\underline{a}})_{\alpha\gamma}, \quad (55)$$

$$(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{a}})_{\beta\gamma} = -6 (\sigma^{\underline{bcde}})_{\gamma}^{\alpha}, \quad (56)$$

$$(\sigma^{\underline{abcde}})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma} = -6 (\sigma^{\underline{bcde}})_{\alpha}^{\gamma}, \quad (57)$$

$$(\sigma^{\underline{ab}}_{\underline{c}})^{\alpha\beta}(\sigma_{\underline{abd}})_{\alpha\gamma} = 7 \cdot 8 (\sigma_{\underline{cd}})_{\gamma}^{\beta} - 8 \cdot 9 \eta_{\underline{cd}} \delta_{\gamma}^{\beta}, \quad (58)$$

$$(\sigma^{\underline{f}}_{\underline{ab}})^{\alpha\beta}(\sigma_{\underline{fcd}})_{\alpha\gamma} = -6 (\sigma_{\underline{bcd}})_{\gamma}^{\beta} - 7 \eta_{\underline{a}[\underline{c}}(\sigma_{\underline{d}]\underline{b}})_{\gamma}^{\beta} - 7 \eta_{\underline{b}[\underline{c}}(\sigma_{\underline{d}]\underline{a}})_{\gamma}^{\beta} + 8 \eta_{\underline{a}[\underline{c}} \eta_{\underline{d}]\underline{b}} \delta_{\gamma}^{\beta}, \quad (59)$$

$$(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{abf}})_{\beta\gamma} = -5 \cdot 6 (\sigma_{\underline{f}}^{\underline{cde}})_{\gamma}^{\alpha} - \frac{6 \cdot 7}{2} \delta_{\underline{f}}^{[\underline{c}} (\sigma^{\underline{de}]}_{\gamma}^{\alpha}, \quad (60)$$

$$(\sigma^{\underline{abc}})^{\alpha\beta}(\sigma_{[\underline{4}]}^{\gamma}(\sigma_{\underline{ab}})_{\gamma}^{\delta} = 4 (\sigma^{\underline{c}})^{\alpha\beta}(\sigma_{[\underline{4}]}^{\delta})_{\beta}^{\alpha} + 4 (\sigma_{[\underline{4}]}^{\alpha})_{\beta}^{\alpha} (\sigma^{\underline{c}})^{\beta\delta}, \quad (61)$$

$$(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{afg}})_{\beta\gamma} = 4 (\sigma_{\underline{fg}}^{\underline{bcde}})_{\gamma}^{\alpha} + \frac{6}{2!} \delta_{[\underline{g}}^{[\underline{d}} \delta_{\underline{f}]}^{\underline{e}} (\sigma^{\underline{bc]}})_{\gamma}^{\alpha} + \frac{5}{3!} \delta_{[\underline{g}}^{[\underline{c}} (\sigma^{\underline{bcd]}})_{\gamma}^{\alpha} \quad (62)$$

$$(\sigma_{\underline{c}}^{\underline{ab}})^{\alpha\beta}(\sigma_{\underline{ad}})_{\beta}^{\gamma}(\sigma_{\underline{eb}})_{\alpha}^{\delta} = 3 \cdot 19 (\sigma_{\underline{cde}})^{\gamma\delta} \quad \text{if } \underline{c} \neq \underline{d} \neq \underline{e}, \quad (63)$$

$$(\sigma_{\underline{ab}}^{\underline{e}})^{\alpha\beta}(\sigma_{\underline{ecd}})_{\alpha\gamma} = 6 (\sigma_{\underline{abcd}})_{\gamma}^{\beta} \quad \text{if } [\underline{ab}] \neq [\underline{cd}], \quad (64)$$

$$(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{af}})_{\beta}^{\gamma}(\sigma_{\underline{bg}})_{\alpha}^{\delta} = 31 (\sigma_{\underline{fg}}^{\underline{cde}})^{\gamma\delta} \quad \text{if } \underline{f} \neq \underline{g} \neq [\underline{cde}] \neq \underline{f}, \quad (65)$$

$$(\sigma^{\underline{ab}}_{\underline{cde}})^{\alpha\beta}(\sigma_{\underline{afg}})_{\beta\gamma} = 4 (\sigma^{\underline{b}}_{\underline{cdefg}})_{\gamma}^{\alpha} - 5 \delta_{[\underline{f}}^{\underline{b}} (\sigma_{\underline{g}]}^{\underline{cde}})_{\gamma}^{\alpha} \quad \text{if } \underline{f} \neq \underline{g} \neq [\underline{cde}], \quad (66)$$

$$(\sigma^{\underline{abc}})_{\alpha\gamma}(\sigma_{\underline{de}})_{\beta}^{\gamma} = -(\sigma^{\underline{abc}}_{\underline{de}})_{\alpha\beta} - \frac{1}{2} \delta_{[\underline{d}}^{[\underline{a}} (\sigma^{\underline{bc]}})_{\underline{e}]}_{\alpha\beta} + \delta_{\underline{d}}^{[\underline{a}} \delta_{\underline{e}}^{\underline{b}} (\sigma^{\underline{c]}})_{\alpha\beta}, \quad (67)$$

$$(\sigma^{\underline{abc}})^{\alpha\gamma}(\sigma_{\underline{de}})_{\gamma}^{\beta} = (\sigma^{\underline{abc}}_{\underline{de}})^{\alpha\beta} + \frac{1}{2} \delta_{[\underline{d}}^{[\underline{a}} (\sigma^{\underline{bc]}})_{\underline{e}]}^{\alpha\beta} - \delta_{\underline{d}}^{[\underline{a}} \delta_{\underline{e}}^{\underline{b}} (\sigma^{\underline{c]}})^{\alpha\beta}, \quad (68)$$

$$(\sigma_{\underline{abcdef}})_{\beta}^{\alpha}(\sigma_{\underline{g}})^{\beta\gamma} = (\sigma_{\underline{abcdefg}})^{\alpha\gamma} - \frac{1}{5!} \eta_{g[\underline{a}} (\sigma_{\underline{bcdef]}})^{\alpha\gamma}, \quad (69)$$

$$(\sigma_{\underline{a}})_{\alpha\beta}(\sigma_{\underline{bcd}})^{\beta\gamma}(\sigma^{\underline{c}})_{\gamma\delta} = -\eta^{\underline{ae}}(\sigma_{\underline{bcd}})_{\alpha\delta} - \frac{1}{2} \delta_{[\underline{b}}^{[\underline{a}} (\sigma_{\underline{cd}]}^{\underline{e}])_{\alpha\delta} + (\sigma^{\underline{ae}}_{\underline{bcd}})_{\alpha\delta} + \delta_{[\underline{b}}^{\underline{a}} \delta_{\underline{c}}^{\underline{e}} (\sigma_{\underline{d}])_{\alpha\delta}, \quad (70)$$

$$(\sigma_{[\underline{3}]}^{\alpha\beta}(\sigma_{\underline{ab}})_{\beta}^{\gamma}(\sigma_{\underline{cd}})_{\alpha}^{\delta} = \frac{1}{3!} \epsilon_{\underline{abcd}[\underline{3}][\underline{3}]} (\sigma^{[\underline{3}]})^{\gamma\delta} \quad \text{if } [\underline{3}] \neq [\underline{a}, \underline{b}, \underline{c}, \underline{d}], \quad (71)$$

$$(\sigma^{\underline{d}})_{(\alpha\beta}(\sigma_{\underline{abcd}})_{\gamma)}^{\delta} = \frac{1}{2} (\sigma_{[\underline{a}})_{(\alpha\beta}(\sigma_{\underline{bcd}]}^{\delta})_{\gamma)}^{\delta}, \quad (72)$$

$$(\sigma_{[\underline{a}})_{(\alpha\beta}(\sigma_{\underline{bcd}]}^{\delta})_{\gamma)}^{\delta} = -(\sigma_{\underline{a}})_{(\alpha\beta}(\sigma^{\underline{d}})^{\delta\epsilon}(\sigma_{\underline{abc}})_{\gamma)}^{\epsilon}, \quad (73)$$

$$(\sigma^{\underline{ab}})_{\alpha}^{\beta}(\sigma_{\underline{c}})_{\beta\gamma}(\sigma_{\underline{ab}})_{\delta}^{\gamma} = 54 (\sigma_{\underline{c}})_{\alpha\delta}, \quad (74)$$

$$(\sigma^{\underline{ab}})_{\alpha}^{\beta}(\sigma_{\underline{cd}})_{\beta}^{\gamma}(\sigma_{\underline{ab}})_{\gamma}^{\delta} = -26 (\sigma^{\underline{cd}})_{\alpha}^{\delta}, \quad (75)$$

$$(\sigma^{\underline{abc}})_{\alpha\gamma}(\sigma_{\underline{dc}})_{\beta}^{\gamma} = 7 (\sigma_{\underline{d}}^{\underline{ab}})_{\alpha\beta} - 8 \delta_{\underline{d}}^{[\underline{a}} (\sigma^{\underline{b]}})_{\alpha\beta}, \quad (76)$$

$$(\sigma_{\underline{ac}})_{\beta}^{\alpha}(\sigma^{\underline{ab}})_{\gamma}^{\beta} = -8 (\sigma^{\underline{b}}_{\underline{c}})_{\gamma}^{\alpha} - 9 \delta_{\underline{c}}^{\underline{b}} \delta_{\gamma}^{\alpha}, \quad (77)$$

$$(\sigma_{\underline{ab}})_{\gamma}^{\beta}(\sigma^{[\underline{4}]}_{\beta})^{\delta}(\sigma^{\underline{b}})_{\delta\alpha} = 2 (\sigma_{\underline{a}})_{\gamma\beta}(\sigma^{[\underline{4}]}_{\alpha})^{\beta} - (\sigma^{[\underline{4}]}_{\gamma})^{\beta}(\sigma_{\underline{a}})_{\beta\alpha}, \quad (78)$$

$$(\sigma^{\underline{bc}})_{\gamma}^{\beta}(\sigma^{[\underline{4}]}_{\beta})^{\delta}(\sigma_{\underline{abc}})_{\delta\alpha} = 2 (\sigma_{\underline{a}})_{\alpha\beta}(\sigma^{[\underline{4}]}_{\gamma})^{\beta} + 4 (\sigma_{\underline{a}})_{\gamma\beta}(\sigma^{[\underline{4}]}_{\alpha})^{\beta}, \quad (79)$$

$$(\sigma^{\underline{abcd}})_{\gamma}^{\beta}(\sigma^{[\underline{4}]}_{\beta})^{\delta}(\sigma_{\underline{bcd}})_{\delta\alpha} = 36 (\sigma^{\underline{a}})_{\gamma\beta}(\sigma^{[\underline{4}]}_{\alpha})^{\beta} - 6 (\sigma^{\underline{a}})_{\alpha\beta}(\sigma^{[\underline{4}]}_{\gamma})^{\beta}, \quad (80)$$

$$(\sigma_{\underline{a}})_{\alpha\beta}(\sigma^{\underline{a}})^{\beta\gamma} = -10 \delta_{\alpha}^{\gamma}, \quad (81)$$

$$(\sigma_{\underline{ab}})_{\alpha}^{\beta}(\sigma^{\underline{ab}})_{\beta}^{\gamma} = -10 \cdot 9 \delta_{\alpha}^{\gamma}, \quad (82)$$

$$(\sigma_{\underline{abc}})_{\alpha\beta}(\sigma^{\underline{abc}})^{\beta\gamma} = 10 \cdot 9 \cdot 8 \delta_{\alpha}^{\gamma}, \quad (83)$$

$$(\sigma_{\underline{abcd}})_{\alpha}^{\beta}(\sigma^{\underline{abcd}})_{\beta}^{\gamma} = 10 \cdot 9 \cdot 8 \cdot 7 \delta_{\alpha}^{\gamma}, \quad (84)$$

$$(\sigma_{\underline{abcde}})_{\alpha\beta}(\sigma^{\underline{abcde}})^{\beta\gamma} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \delta_{\alpha}^{\gamma}, \quad (85)$$

$$(\sigma^{\underline{f}})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma}(\sigma_{\underline{f}})_{\gamma\delta} = 8 (\sigma_{\underline{a}})_{\alpha\delta}, \quad (86)$$

$$(\sigma^{\underline{f}})_{\alpha\beta}(\sigma_{\underline{a}})_{\beta\gamma}(\sigma_{\underline{f}})^{\gamma\delta} = 8 (\sigma_{\underline{a}})^{\alpha\delta}, \quad (87)$$

$$(\sigma^{\underline{f}})_{\alpha\beta}(\sigma_{\underline{ab}})_{\gamma}^{\beta}(\sigma_{\underline{f}})^{\gamma\delta} = 6 (\sigma_{\underline{ab}})_{\alpha}^{\delta}, \quad (88)$$

$$(\sigma^{\underline{f}})_{\alpha\beta}(\sigma_{\underline{abc}})^{\beta\gamma}(\sigma_{\underline{f}})_{\gamma\delta} = 4 (\sigma_{\underline{abc}})_{\alpha\delta}, \quad (89)$$

$$(\sigma_{\underline{f}})^{\alpha\beta}(\sigma_{\underline{abc}})_{\beta\gamma}(\sigma_{\underline{f}})^{\gamma\delta} = 4 (\sigma_{\underline{abc}})^{\alpha\delta}, \quad (90)$$

$$(\sigma_{\underline{f}})^{\alpha\beta}(\sigma_{\underline{abcd}})_{\gamma}^{\beta}(\sigma_{\underline{f}})^{\gamma\delta} = -2 (\sigma_{\underline{abcd}})_{\alpha}^{\delta}, \quad (91)$$

$$(\sigma_{\underline{f}})^{\alpha\beta}(\sigma_{[5]})^{\beta\gamma}(\sigma_{\underline{f}})_{\gamma\delta} = 0, \quad (92)$$

$$(\sigma_{\underline{f}_1\underline{f}_2})_{\alpha}^{\beta}(\sigma_{\underline{a}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}^{\gamma} = 54 (\sigma_{\underline{a}})_{\alpha\delta}, \quad (93)$$

$$(\sigma_{\underline{f}_1\underline{f}_2})_{\alpha}^{\beta}(\sigma_{\underline{a}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\gamma}^{\delta} = 54 (\sigma_{\underline{a}})^{\beta\delta}, \quad (94)$$

$$(\sigma_{\underline{f}_1\underline{f}_2})_{\alpha}^{\beta}(\sigma_{\underline{ab}})_{\gamma}^{\alpha}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}^{\gamma} = -26 (\sigma_{\underline{ab}})_{\delta}^{\beta}, \quad (95)$$

$$(\sigma_{\underline{f}_1\underline{f}_2})_{\alpha}^{\beta}(\sigma_{\underline{abc}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}^{\gamma} = 6 (\sigma_{\underline{abc}})_{\alpha\delta}, \quad (96)$$

$$(\sigma_{\underline{f}_1\underline{f}_2})_{\alpha}^{\beta}(\sigma_{\underline{abc}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\gamma}^{\delta} = 6 (\sigma_{\underline{abc}})^{\beta\delta}, \quad (97)$$

$$(\sigma_{\underline{f}_1\underline{f}_2})_{\alpha}^{\beta}(\sigma_{\underline{abcd}})_{\gamma}^{\alpha}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}^{\gamma} = 6 (\sigma_{\underline{abcd}})_{\delta}^{\beta}, \quad (98)$$

$$(\sigma_{\underline{f}_1\underline{f}_2})_{\alpha}^{\beta}(\sigma_{\underline{abcde}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}^{\gamma} = -10 (\sigma_{\underline{abcde}})_{\alpha\delta}, \quad (99)$$

$$(\sigma_{\underline{f}_1\underline{f}_2})_{\alpha}^{\beta}(\sigma_{\underline{abcde}})^{\alpha\delta}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}^{\gamma} = -10 (\sigma_{\underline{abcde}})^{\beta\gamma}, \quad (100)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\gamma\delta} = -8 \cdot 36 (\sigma_{\underline{a}})_{\alpha\delta}, \quad (101)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{a}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\gamma\delta} = -8 \cdot 36 (\sigma_{\underline{a}})^{\beta\delta}, \quad (102)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{ab}})_{\gamma}^{\beta}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\delta}^{\gamma} = -48 (\sigma_{\underline{ab}})_{\alpha}^{\delta}, \quad (103)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{abc}})^{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\gamma\delta} = 48 (\sigma_{\underline{abc}})_{\alpha\delta}, \quad (104)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{abc}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\gamma\delta} = 48 (\sigma_{\underline{abc}})^{\beta\delta}, \quad (105)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{abcd}})_{\gamma}^{\beta}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\delta}^{\gamma} = -48 (\sigma_{\underline{abcd}})_{\alpha}^{\delta}, \quad (106)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{[5]})^{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\gamma\delta} = 0, \quad (107)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}^{\beta}(\sigma_{\underline{a}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}^{\gamma} = 24 \cdot 42 (\sigma_{\underline{a}})_{\alpha\delta}, \quad (108)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}^{\beta}(\sigma_{\underline{a}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}^{\gamma} = 24 \cdot 42 (\sigma_{\underline{a}})^{\beta\delta}, \quad (109)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}^{\beta}(\sigma_{\underline{ab}})_{\gamma}^{\alpha}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}^{\gamma} = -24 \cdot 14 (\sigma_{\underline{ab}})_{\delta}^{\beta}, \quad (110)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}^{\beta}(\sigma_{\underline{abc}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}^{\gamma} = -24 \cdot 14 (\sigma_{\underline{abc}})_{\alpha\delta}, \quad (111)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}^{\beta}(\sigma_{\underline{abc}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\gamma}^{\delta} = -24 \cdot 14 (\sigma_{\underline{abc}})^{\beta\delta}, \quad (112)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}^{\beta}(\sigma_{\underline{abcd}})_{\gamma}^{\alpha}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}^{\gamma} = 48 (\sigma_{\underline{abcd}})_{\delta}^{\beta}, \quad (113)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}^{\beta}(\sigma_{\underline{abcde}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}^{\gamma} = 240 (\sigma_{\underline{abcde}})_{\alpha\delta}, \quad (114)$$

$$(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}^{\beta}(\sigma_{\underline{abcde}})^{\alpha\delta}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}^{\gamma} = 240 (\sigma_{\underline{abcde}})^{\beta\gamma}. \quad (115)$$

IV. THE PROGRAM

(*BeginPackage["SigmaVector"]*)

Print["This is SigmaVector program. It reduces the product of sigma matrices.

Spinor indices of the chiral sigma matrices analyzed here are suppressed.

Sigma matrices with spinor indices down are denoted by S, as well as those

matrices which have one spinor index down and other up. If both spinor indices

are up, the letter SB is used.

The program handles all contraction possibilities of spinor indices, except when two sigma matrices, both with even number of vector indices are multiplied in such a way that the lower index of the first sigma matrix is summed with the upper index of the second matrix. In this case the order of sigma matrices should be reversed.

Here is an example:

$$(\backslash s^{\{e, f\}})_{-}\{a\}^{\{b\}} * (\backslash s_{-}\{a, b, c\})^{\{a\}g} * (\backslash s_{-}\{e, f\})_{-}\{g\}^{\{d\}}$$

should be represented in the symbolic form as:

*ExpandAll[ExpandAll[S[e, f] ** SB[a, b, c]] ** S[e, f] /. Rule1],*

where Rule1 is a rule which transforms sigma matrices with 6 or more vector indices in appropriate sigma matrices with 4 or less vector indices and contracts corresponding epsilon symbols.”]

(* These next rules assumed repeated indices for implied sum *)

SetAttributes[Id, Orderless]

Ten=10

SetAttributes[Eta, Orderless]

Eta/: Eta[a_, b_] ^ 2 := Ten (* 10 for our purposes *)

Eta/: Eta[a_, a_] := Ten

Eta/: Eta[a_, b_] * S[d___, a_, c___] := S[d, b, c]

Eta/: S[d___, a_, c___] * Eta[a_, b_] := S[d, b, c]

Eta/: Eta[a_, b_] * SB[d___, a_, c___] := SB[d, b, c]

Eta/: SB[d___, a_, c___] * Eta[a_, b_] := SB[d, b, c]

Eta/: Eta[a_, b_] * Epsilon[d___, a_, c___] := Epsilon[d, b, c]

Eta/: Epsilon[d___, a_, c___] * Eta[a_, b_] := Epsilon[d, b, c]

Eta/: Eta[a_, b_] * Eta[a_, c_] := Eta[b, c]

Eta/: Eta[a_, c_] * Eta[a_, b_] := Eta[b, c]

Eta/: NumberQ[Eta[x___]] := True

(* These rules define Ten as the dimension so that its dependence in the

calculations can be followed. *)

Unprotect[NumberQ]

NumberQ[Ten^n_Integer] := True

NumberQ[Ten] := True

```

NumberQ[1/four!]:=True    (* used to denote 1/4! *)
NumberQ[1/three!]:=True   (* used to denote 1/3! *)
Protect[NumberQ]
(* Set the tag parameter for implied sums here *)
epi = 1
reset:=(epi=1)
Par[exp_]:= (exp)
(* This comes from Epsilon.m *)
(* Rules for Epsilon when Euclidean space metric is present *)
(* These Rules are true for any dimension. Just change Ten to
the appropriate dimension. See also Rule2. *)
Epsilon/: NumberQ[Epsilon[x_]]:=True
Epsilon/: Epsilon[b_]*Epsilon[b_] := Ten!/; Signature[{b}]!=0
Epsilon/: Epsilon[b_]^2 := Ten!
Unprotect[NonCommutativeMultiply]
(x_?NumberQ a_[i_])**(y_?NumberQ b_[j_]):=(x y) a[i]**b[j]
(a_+b_)**c_:= a**c +b**c
a_**(b_+c_):= a**b + a**c
a_**(x_?NumberQ b_):= x a**b
(x_?NumberQ a_)**b_:= x a**b
b_?NumberQ ** a_:= b a
a_** b_?NumberQ:= b a
((anything_)*c_)**d_:= (anything)(c**d)
Protect[NonCommutativeMultiply]
(* Definition of an antisymmetrizer. *)
SetAttributes[AntiSymmetrize, HoldAll]
AntiSymmetrize/: AntiSymmetrize[ k_, exp_] :=
  Sum[Signature[Permutations[k][[i]]]*Signature[k]*
    (exp /. Table[k[[j]] -> Permutations[k][[i]][[j]],
      {j, Length[k]})),
    {i, 1, Length[k]}]
(* These rules define S,SB and Epsilon as purely
anti-symmetric tensors

```



```

for any rank. *)
S[x_] := Signature[{x}] S@@Sort[{x}] /; Not[OrderedQ[{x}]]
SB[x_] := Signature[{x}] SB@@Sort[{x}] /; Not[OrderedQ[{x}]]
Epsilon[x_] := Signature[{x}] Epsilon@@Sort[{x}] /;
Not[OrderedQ[{x}]]
S[x_]:=0 /; Signature[{x}]==0
SB[x_]:=0 /; Signature[{x}]==0
Epsilon[x_]:=0 /; Signature[{x}]==0
(* Epsilon[1,2,3,4,5,6,7,8,9,10] = 1 *)
(* This rule replaces 6,7,8,9, and 10 forms with their duals *)
Rule1 = {
S[a_,b_,c_,d_,e_,f_] :>
( Epsilon[a,b,c,d,e,f,$1[epi],$2[epi],$3[epi],$4[epi]]*
S[$1[epi],$2[epi],$3[epi],$4[epi++]])/24,
S[d[1],d[2],d[3],d[4],d[5],d[6]]:>
(Epsilon[d[1],d[2],d[3],d[4],d[5],d[6],$1[epi],$2[epi],$3[epi],$4[epi]]*
S[$1[epi],$2[epi],$3[epi],$4[epi++]])/24,
SB[a_,b_,c_,d_,e_,f_,g_] :>
(- Epsilon[a,b,c,d,e,f,g,$1[epi],$2[epi],$3[epi]]*
SB[$1[epi],$2[epi],$3[epi++]])/6,
SB[d[1],d[2],d[3],d[4],d[5],d[6],d[7]]:>
(-Epsilon[d[1],d[2],d[3],d[4],d[5],d[6],d[7],1[epi],$2[epi],$3[epi]]*
SB[$1[epi],$2[epi],$3[epi++]])/6,
S[a_,b_,c_,d_,e_,f_,g_] :>
(- Epsilon[a,b,c,d,e,f,g,$1[epi],$2[epi],$3[epi]]*
S[$1[epi],$2[epi],$3[epi++]])/6,
S[d[1],d[2],d[3],d[4],d[5],d[6],d[7]]:>
(-Epsilon[d[1],d[2],d[3],d[4],d[5],d[6],d[7],$1[epi],$2[epi],$3[epi]]*
S[$1[epi],$2[epi],$3[epi++]])/6,

S[a_,b_,c_,d_,e_,f_,g_,h_] :>
(-Epsilon[a,b,c,d,e,f,g,h,$1[epi],$2[epi]]*
S[$1[epi],$2[epi++]])/2,

```

```

S[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8]] :>
(-Epsilon[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8],$1[epi],$2[epi]]*
S[$1[epi],$2[epi++]])/2,
SB[a_,b_,c_,d_,e_,f_,g_,h_,i_] :>
(Epsilon[a,b,c,d,e,f,g,h,i,$1[epi]]*
SB[$1[epi++]]) ,
SB[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8],d[9]] :>
(Epsilon[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8],d[9],$1[epi]]*
SB[$1[epi++]]) ,
S[a_,b_,c_,d_,e_,f_,g_,h_,i_] :>
(Epsilon[a,b,c,d,e,f,g,h,i,$1[epi]]*
S[$1[epi++]])
}
(* This rule pulls out the coefficients of chiral matrices. *)
SetAttributes[Eta2,Orderless]
Eta2/: NumberQ[Eta2[x_]] := True
Rule2 = {
S[a_] :> S[d[1]] * Eta2[d[1],a],
S[a_,b_] :> S[d[1],d[2]] * Eta2[d[1],a] Eta2[d[2],b],
S[a_,b_,c_] :> S[d[1],d[2],d[3]]*Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c],
S[a_,b_,c_,d1_] :> S[d[1],d[2],d[3],d[4]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1],
S[a_,b_,c_,d1_,e_] :>S[d[1],d[2],d[3],d[4],d[5]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1] Eta2[d[5],e],
S[a_,b_,c_,d1_,e_,f_] :>S[d[1],d[2],d[3],d[4],d[5],d[6]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1] Eta2[d[5],e] *
Eta2[f,d[6]],
S[a_,b_,c_,d1_,e_,f_,g_] :>S[d[1],d[2],d[3],d[4],d[5],d[6],d[7]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1] Eta2[d[5],e] *
Eta2[f,d[6]] Eta2[g,d[7]],
S[a_,b_,c_,d1_,e_,f_,g_,h_] :>S[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1] Eta2[d[5],e] *
Eta2[f,d[6]] Eta2[g,d[7]] Eta2[h,d[8]],

```

```

S[a_,b_,c_,d1_,e_,f_,g_,h_,i_] :>S[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8],d[9]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1] Eta2[d[5],e] *
Eta2[f,d[6]] Eta2[g,d[7]] Eta2[h,d[8]] Eta2[i,d[9]],
S[a_,b_,c_,d1_,e_,f_,g_,h_,i_,j_] :>S[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8],d[9],d[10]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1] Eta2[d[5],e] *
Eta2[f,d[6]] Eta2[g,d[7]] Eta2[h,d[8]] Eta2[i,d[9]] Eta2[j,d[10]],
SB[a_] :> SB[d[1]] * Eta2[d[1],a],
SB[a_,b_,c_] :> SB[d[1],d[2],d[3]]*Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c],
SB[a_,b_,c_,d1_,e_] :>SB[d[1],d[2],d[3],d[4],d[5]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1] Eta2[d[5],e],
SB[a_,b_,c_,d1_,e_,f_,g_] :>SB[d[1],d[2],d[3],d[4],d[5],d[6],d[7]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1] Eta2[d[5],e] *
Eta2[f,d[6]] Eta2[g,d[7]],
SB[a_,b_,c_,d1_,e_,f_,g_,h_,i_] :>SB[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8],d[9]] *
Eta2[d[1],a]* Eta2[d[2],b] Eta2[d[3],c] Eta2[d[4],d1] Eta2[d[5],e] *
Eta2[f,d[6]] Eta2[g,d[7]] Eta2[h,d[8]] Eta2[i,d[9]]
}
(* Use Recover to factor out the S's and SB's *)
Recover[exp_] := Collect[exp/.Rule2,{ S[d[1]], S[d[1],d[2]],S[d[1],d[2],d[3]],
S[d[1],d[2],d[3],d[4]], S[d[1],d[2],d[3],d[4],d[5]],
S[d[1],d[2],d[3],d[4],d[5],d[6]],
S[d[1],d[2],d[3],d[4],d[5],d[6],d[7]],
S[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8]],
S[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8],d[9]],
S[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8],d[9],d[10]],
SB[d[1]], SB[d[1],d[2],d[3]], SB[d[1],d[2],d[3],d[4],d[5]],
SB[d[1],d[2],d[3],d[4],d[5],d[6],d[7]],
SB[d[1],d[2],d[3],d[4],d[5],d[6],d[7],d[8],d[9]]}]/.Rule3
Rule3={Eta2[f_] :> Eta[f]}

(* This rule is time consuming. *)
Epsilon/: Epsilon[b_] * Epsilon[c_] :=
-Par[ Signature[ Join[Complement[{b},

```

```

Intersection[{b},{c}],
Intersection[{b},{c}]]*
Signature[ Join[Complement[{c},Intersection[{b},{c}]],
Intersection[{b},{c}]]]*ReleaseHold[ (
  AntiSymmetrize[Complement[{c}, Intersection[{b}, {c}]],
  Product[Eta[Complement[{b}, Intersection[{b}, {c}]]][[i]],
    Complement[{c}, Intersection[{b}, {c}]]][[i]],
    {i, Length[Complement[{c}, Intersection[{b},
    {c}]]]}]])]*(Length[
  Intersection[{b}, {c}]]!) /; {b}!= {c} && Length[{b}]==Length[{c}]
S/: S[i_]**SB[j_] := - Eta[i, j] - S[i, j]
S/: S[i_]**S[j_,k_] := -S[k] Eta[i, j] + S[j] Eta[i, k] -
S[i, j, k]
S/: S[j_,k_]**S[i_] := -S[k] Eta[i, j] + S[j] Eta[i, k] +
S[i, j, k]
S/: S[j_,k_]**SB[i_] := Eta[i,j] SB[k] - Eta[i,k] SB[j] -
SB[i,j,k]
S/: S[i_] ** SB[j_, k_, l_] :=
-S[k, l] Eta[i, j] - S[l,j] Eta[i, k] - S[j, k] Eta[i, l] -
S[i, j, k, l]
S/: S[j_,k_,l_]**SB[i_] := -Eta[i,j] S[k,l] - Eta[i,k] S[l,j] -
Eta[i,l] S[j,k] + S[i,j,k,l]
S/: S[i_]**S[j_,k_,l_,m_] := Eta[i,j] S[k,l,m] -
Eta[i,k] S[l,m,j] + Eta[i,l] S[j,k,m] -
Eta[i,m] S[j,k,l] + S[i,j,k,l,m]
S/: S[j_,k_,l_,m_]**S[i_] := -Eta[i,j] S[k,l,m] +
Eta[i,k] S[l,m,j] - Eta[i,l] S[j,k,m] +
Eta[i,m] S[j,k,l] + S[i,j,k,l,m]
S/: S[j_,k_,l_,m_]**SB[i_] := -Eta[i,j] SB[k,l,m] +
Eta[i,k] SB[l,m,j] - Eta[i,l] SB[j,k,m] +
Eta[i,m] SB[j,k,l] + SB[i,j,k,l,m]
S/: S[i_]**SB[j_,k_,l_,m_,n_] := -Eta[i,j] S[k,l,m,n] -
Eta[i,k] S[l,m,n,j] - Eta[i,l] S[m,n,j,k] -

```

$$\begin{aligned}
& \text{Eta}[i,m] \text{ S}[n,j,k,l] - \text{Eta}[i,n] \text{ S}[j,k,l,m] - \text{S}[i,j,k,l,m,n] \\
& \text{S}/: \text{S}[j_-,k_-,l_-,m_-,n_]**\text{SB}[i_]:= -\text{Eta}[i,j] \text{ S}[k,l,m,n] - \\
& \text{Eta}[i,k] \text{ S}[l,m,n,j] - \text{Eta}[i,l] \text{ S}[m,n,j,k] - \\
& \text{Eta}[i,m] \text{ S}[n,j,k,l] - \text{Eta}[i,n] \text{ S}[j,k,l,m] + \text{S}[i,j,k,l,m,n] \\
& \text{S}/: \text{S}[i_]**\text{S}[j_-,k_-,l_-,m_-,n_-,p_]:= -\text{S}[i,j,k,l,m,n,p] + \\
& (\text{Epsilon}[j,k,l,m,n,p,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}], \$4[\text{epi}]]* \\
& \text{S}[i,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}], \$4[\text{epi}++]])/24 \\
& \text{S}/: \text{S}[j_-,k_-,l_-,m_-,n_-,p_]**\text{SB}[i_]:= \text{SB}[i,j,k,l,m,n,p] + \\
& (\text{Epsilon}[j,k,l,m,n,p,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}], \$4[\text{epi}]]* \\
& \text{SB}[i,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}], \$4[\text{epi}++]])/24 \\
& \text{S}/: \text{S}[j_-,k_-,l_-,m_-,n_-,p_]**\text{S}[i_]:= \text{S}[i,j,k,l,m,n,p] + \\
& (\text{Epsilon}[j,k,l,m,n,p,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}], \$4[\text{epi}]]* \\
& \text{S}[i,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}], \$4[\text{epi}++]])/24 \\
& \text{S}/: \text{S}[i_]**\text{SB}[j_-,k_-,l_-,m_-,n_-,p_-,q_]:= \text{S}[i,j,k,l,m,n,p,q] + \\
& (\text{Epsilon}[j,k,l,m,n,p,q,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}]]* \\
& \text{S}[i,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}++]])/6 \\
& \text{S}/: \text{S}[j_-,k_-,l_-,m_-,n_-,p_-,q_]**\text{SB}[i_]:= \text{S}[i,j,k,l,m,n,p,q] - \\
& (\text{Epsilon}[j,k,l,m,n,p,q,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}]]* \\
& \text{S}[i,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}++]])/6 \\
& \text{S}/: \text{S}[i_]**\text{S}[j_-,k_-,l_-,m_-,n_-,o_-,p_-,q_]:= \\
& \text{S}[i,j,k,l,m,n,o,p,q] + \\
& (\text{Epsilon}[j,k,l,m,n,o,p,q,\$1[\text{epi}], \$2[\text{epi}]]* \\
& \text{S}[i,\$1[\text{epi}], \$2[\text{epi}++]])/2 \\
& \text{S}/: \text{S}[j_-,k_-,l_-,m_-,n_-,o_-,p_-,q_]**\text{S}[i_]:= \\
& \text{S}[i,j,k,l,m,n,o,p,q] - \\
& (\text{Epsilon}[j,k,l,m,n,o,p,q,\$1[\text{epi}], \$2[\text{epi}]]* \\
& \text{S}[i,\$1[\text{epi}], \$2[\text{epi}++]])/2 \\
& \text{S}/: \text{S}[j_-,k_-,l_-,m_-,n_-,o_-,p_-,q_]**\text{SB}[i_]:= \\
& -\text{SB}[i,j,k,l,m,n,o,p,q] + \\
& (\text{Epsilon}[j,k,l,m,n,o,p,q,\$1[\text{epi}], \$2[\text{epi}]]* \\
& \text{SB}[i,\$1[\text{epi}], \$2[\text{epi}++]])/2 \\
& \text{S}/: \text{S}[i_]**\text{SB}[j_-,k_-,l_-,m_-,n_-,o_-,p_-,q_-,r_]:= \\
& \text{Epsilon}[i,j,k,l,m,n,o,p,q,r] -
\end{aligned}$$

$\text{Epsilon}[j,k,l,m,n,o,p,q,r,\$1[\text{epi}]] * S[i,\$1[\text{epi}++]]$
 $S/: S[j_k_l_m_n_o_p_q_r_]**SB[i_]:=$
 $\text{Epsilon}[i,j,k,l,m,n,o,p,q,r] +$
 $\text{Epsilon}[j,k,l,m,n,o,p,q,r,\$1[\text{epi}]] * S[i,\$1[\text{epi}++]]$
 $SB/: SB[i_]**S[j_]:= -\text{Eta}[i,j] + S[i,j]$
 $SB/: SB[i_]**S[j_k_]:= \text{Eta}[i,j] SB[k] - \text{Eta}[i,k] SB[j] +$
 $SB[i,j,k]$
 $SB/: SB[i_]**S[j_k_l_]:= \text{Eta}[i,j] S[k,l] +$
 $\text{Eta}[i,k] S[l,j] + \text{Eta}[i,l] S[j,k] - S[i,j,k,l]$
 $SB/: SB[j_k_l_]**S[i_]:=$
 $S[k,l] \text{Eta}[i,j] + S[l,j] \text{Eta}[i,k] + S[j,k] \text{Eta}[i,l] +$
 $S[i,j,k,l]$
 $SB/: SB[i_]**S[j_k_l_m_]:= \text{Eta}[i,j] SB[k,l,m] -$
 $\text{Eta}[i,k] SB[l,m,j] + \text{Eta}[i,l] SB[j,k,m] -$
 $\text{Eta}[i,m] SB[j,k,l] + SB[i,j,k,l,m]$
 $SB/: SB[i_]**S[j_k_l_m_n_]:= -\text{Eta}[i,j] S[k,l,m,n] -$
 $\text{Eta}[i,k] S[l,m,n,j] - \text{Eta}[i,l] S[m,n,j,k] -$
 $\text{Eta}[i,m] S[n,j,k,l] - \text{Eta}[i,n] S[j,k,l,m] + S[i,j,k,l,m,n]$
 $SB/: SB[j_k_l_m_n_]**S[i_]:= -\text{Eta}[i,j] S[k,l,m,n] -$
 $\text{Eta}[i,k] S[l,m,n,j] - \text{Eta}[i,l] S[m,n,j,k] -$
 $\text{Eta}[i,m] S[n,j,k,l] - \text{Eta}[i,n] S[j,k,l,m] - S[i,j,k,l,m,n]$
 $SB/: SB[i_]**S[j_k_l_m_n_p_]:= -SB[i,j,k,l,m,n,p] +$
 $(\text{Epsilon}[j,k,l,m,n,p,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}], \$4[\text{epi}]] * S[i,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}], \$4[\text{epi}++]])/24$
 $SB/: SB[j_k_l_m_n_p_q_]**S[i_]:= -S[i,j,k,l,m,n,p,q] -$
 $(\text{Epsilon}[j,k,l,m,n,p,q,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}]] * S[i,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}++]])/6$
 $SB/: SB[i_]**S[j_k_l_m_n_p_q_]:= -S[i,j,k,l,m,n,p,q] +$
 $(\text{Epsilon}[j,k,l,m,n,p,q,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}]] * S[i,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}++]])/6$
 $SB/: SB[i_]**S[j_k_l_m_n_o_p_q_]:=$
 $-SB[i,j,k,l,m,n,o,p,q] -$
 $(\text{Epsilon}[j,k,l,m,n,o,p,q,\$1[\text{epi}], \$2[\text{epi}]] * S[i,\$1[\text{epi}], \$2[\text{epi}], \$3[\text{epi}++]])/6$

$SB[i, \$1[epi], \$2[epi++]]/2$
 $SB/: SB[j, k, l, m, n, o, p, q, r] ** S[i] :=$
 $Epsilon[i, j, k, l, m, n, o, p, q, r] -$
 $Epsilon[j, k, l, m, n, o, p, q, r, \$1[epi]] * S[i, \$1[epi++]]$
 $SB/: SB[i] ** S[j, k, l, m, n, o, p, q, r] :=$
 $Epsilon[i, j, k, l, m, n, o, p, q, r] +$
 $Epsilon[j, k, l, m, n, o, p, q, r, \$1[epi]] * S[i, \$1[epi++]]$
 $S/: S[i, j] ** S[k, l] := -Eta[k, l] S[i, j] -$
 $Expand[Par[S[i, j] ** S[k]] ** SB[l]]$
 $S/: S[a, b, c, d] ** S[k, l] :=$
 $-Eta[k, l] S[a, b, c, d] -$
 $Expand[Par[S[a, b, c, d] ** S[k]] ** SB[l]]$
 $S/: S[a, b, c, d, e, f] ** S[k, l] :=$
 $-Eta[k, l] S[a, b, c, d, e, f] -$
 $Expand[Par[S[a, b, c, d, e, f] ** S[k]] ** SB[l]]$
 $S/: S[a, b, c, d, e, f, g, h] ** S[k, l] :=$
 $-Eta[k, l] S[a, b, c, d, e, f, g, h] -$
 $Expand[Par[S[a, b, c, d, e, f, g, h] ** S[k]] ** SB[l]]$
 $S/: S[a, b, c] ** S[k, l] :=$
 $Eta[k, l] S[a, b, c] +$
 $Expand[Par[S[a, b, c] ** SB[k]] ** S[l]]$
 $S/: S[a, b, c, d, e] ** S[k, l] :=$
 $Eta[k, l] S[a, b, c, d, e] +$
 $Expand[Par[S[a, b, c, d, e] ** SB[k]] ** S[l]]$
 $S/: S[a, b, c, d, e, f, g] ** S[k, l] :=$
 $Eta[k, l] S[a, b, c, d, e, f, g] +$
 $Expand[Par[S[a, b, c, d, e, f, g] ** SB[k]] ** S[l]]$
 $S/: S[a, b, c, d, e, f, g, h, m] ** S[k, l] :=$
 $Eta[k, l] S[a, b, c, d, e, f, g, h, m] +$
 $Expand[Par[S[a, b, c, d, e, f, g, h, m] ** SB[k]] ** S[l]]$
 $S/: S[i, j] ** S[k, l] := -Eta[i, j] S[k, l] -$
 $Expand[S[i] ** Par[SB[j] ** S[k, l]]]$
 $S/: S[i, j] ** SB[k, l] := Eta[i, j] SB[k, l] +$

```

Expand[ SB[i]**Par[S[j]**SB[k,l]]
SB/: SB[i_,j_]**S[k_,l_] := - Eta[k,l] SB[i,j] -
Expand[Par[SB[i,j]**S[k]**SB[l]]
S/: S[i_,j_,k_]**S[l_,m_,n_] :=
- Expand[S[i]**Par[SB[j]**Par[S[k]**S[l,m,n]]]] -
Eta[i,j] S[k]**S[l,m,n] + Eta[i,k] S[j]**S[l,m,n]-
Eta[j,k] S[i]**S[l,m,n]
S/: S[i_,j_,k_]**SB[l_,m_,n_] :=
- Expand[S[i]**Par[SB[j]**
Par[S[k]**SB[l,m,n]]]] -
Eta[i,j] S[k]**SB[l,m,n] + Eta[i,k] S[j]**SB[l,m,n] -
Eta[j,k] S[i]**SB[l,m,n]
S/: S[i_,j_,k_]**S[l_,m_,n_] :=
- Expand[Par[Par[S[i,j,k]**S[l]]**SB[m]**
S[n]] -
Eta[l,m] S[i,j,k]**S[n] + Eta[l,n] S[i,j,k]**S[m] -
Eta[m,n] S[i,j,k]**S[l]
S/: S[a_,b_,c_]**SB[i_,j_,k_] :=
- Expand[Par[Par[S[a,b,c]**SB[j]]**S[k]**
SB[i]] -
Eta[j,k] S[a,b,c]**SB[i] + Eta[i,j] S[a,b,c]**SB[k] -
Eta[i,k] S[a,b,c]**SB[j]
S/: SB[a_,b_,c_]**S[i_,j_,k_] :=
- Expand[Par[Par[SB[a,b,c]**S[i]]**SB[j]]**S[k]] -
Eta[j,k] SB[a,b,c]**S[i] - Eta[i,j] SB[a,b,c]**S[k] +
Eta[i,k] SB[a,b,c]**S[j]
SB/: SB[i_,j_,k_]**S[l_,m_,n_] :=
- Expand[SB[j]**Par[S[k]**
Par[SB[i]**S[l,m,n]]]] +
Eta[i,j] SB[k]**S[l,m,n] - Eta[i,k] SB[j]**S[l,m,n] -
Eta[j,k] SB[i]**S[l,m,n]
S/: S[a_,b_,c_,d_]**S[i_,j_,k_,l_] :=
Expand[S[b,c,d]**Par[SB[a]**S[i,j,k,l]]]+

```



```

Eta[a,b] Par[S[c,d]**S[i,j,k,l]]-
Eta[a,c] Par[S[b,d]**S[i,j,k,l]]+
Eta[a,d] Par[S[b,c]**S[i,j,k,l]]
S/: S[a_,b_,c_,d_]**SB[i_,j_,k_,l_] :=
- Expand[SB[a]**Par[S[b,c,d]**SB[i,j,k,l]]]+
Eta[a,b] Par[S[c,d]**SB[i,j,k,l]]-
Eta[a,c] Par[S[b,d]**SB[i,j,k,l]]+
Eta[a,d] Par[S[b,c]**SB[i,j,k,l]]
S/: S[a_,b_,c_,d_]**S[i_,j_,k_,l_] :=
- Expand[Par[S[a,b,c,d]**S[i]]**SB[j,k,l]]-
Eta[i,j] Par[S[a,b,c,d]**S[k,l]]+
Eta[i,k] Par[S[a,b,c,d]**S[j,l]]-
Eta[i,l] Par[S[a,b,c,d]**S[j,k]]
S/: S[a_,b_,c_,d_,e_,f_]**S[i_,j_,k_,l_] :=
- Expand[Par[S[a,b,c,d,e,f]**S[i]]**SB[j,k,l]]-
Eta[i,j] Par[S[a,b,c,d,e,f]**S[k,l]]+
Eta[i,k] Par[S[a,b,c,d,e,f]**S[j,l]]-
Eta[i,l] Par[S[a,b,c,d,e,f]**S[j,k]]
S/: S[a_,b_,c_,d_,e_,f_,g_,h_]**S[i_,j_,k_,l_] :=
- Expand[Par[S[a,b,c,d,e,f,g,h]**S[i]]**SB[j,k,l]]-
Eta[i,j] Par[S[a,b,c,d,e,f,g,h]**S[k,l]]+
Eta[i,k] Par[S[a,b,c,d,e,f,g,h]**S[j,l]]-
Eta[i,l] Par[S[a,b,c,d,e,f,g,h]**S[j,k]]
S/: S[a_,b_,c_,d_,e_]**S[i_,j_,k_,l_] :=
- Expand[
Par[S[a,b,c,d,e]**SB[i]]**S[j,k,l]]+
Eta[i,j] Par[S[a,b,c,d,e]**S[k,l]]-
Eta[i,k] Par[S[a,b,c,d,e]**S[j,l]]+
Eta[i,l] Par[S[a,b,c,d,e]**S[j,k]]
S/: S[a_,b_,c_,d_,e_,f_,g_]**S[i_,j_,k_,l_] :=
- Expand[
Par[S[a,b,c,d,e,f,g]**SB[i]]**S[j,k,l]]+
Eta[i,j] Par[S[a,b,c,d,e,f,g]**S[k,l]]-

```

```

Eta[i,k] Par[S[a,b,c,d,e,f,g]**S[j,l]]+
Eta[i,l] Par[S[a,b,c,d,e,f,g]**S[j,k]]
S/: S[a_,b_,c_,d_,e_,f_,g_,h_,m_]**S[i_,j_,k_,l_] :=
- Expand[
Par[S[a,b,c,d,e,f,g,h,m]**SB[i]**S[j,k,l]]+
Eta[i,j] Par[S[a,b,c,d,e,f,g,h,m]**S[k,l]]-
Eta[i,k] Par[S[a,b,c,d,e,f,g,h,m]**S[j,l]]+
Eta[i,l] Par[S[a,b,c,d,e,f,g,h,m]**S[j,k]]
S/: SB[a_,b_,c_,d_]**S[i_,j_,k_,l_] :=
- Expand[Par[SB[a,b,c,d]**S[i]**SB[j,k,l]]-
Eta[i,j] Par[SB[a,b,c,d]**S[k,l]]+
Eta[i,k] Par[SB[a,b,c,d]**S[j,l]]-
Eta[i,l] Par[SB[a,b,c,d]**S[j,k]]
S/: S[a_,b_,c_,d_,e_]**SB[i_,j_,k_,l_,m_] :=
Expand[S[a]**Par[S[b,c,d,e]**SB[i,j,k,l,m]]]-
Eta[a,b] Par[S[c,d,e]**SB[i,j,k,l,m]] +
Eta[a,c] Par[S[b,d,e]**SB[i,j,k,l,m]] -
Eta[a,d] Par[S[b,c,e]**SB[i,j,k,l,m]] +
Eta[a,e] Par[S[b,c,d]**SB[i,j,k,l,m]]
SB/: SB[a_,b_,c_,d_,e_]**S[i_,j_,k_,l_,m_] :=
Expand[S[b,c,d,e]**Par[SB[a]**S[i,j,k,l,m]]]+
Eta[a,b] Par[SB[c,d,e]**S[i,j,k,l,m]] -
Eta[a,c] Par[SB[b,d,e]**S[i,j,k,l,m]] +
Eta[a,d] Par[SB[b,c,e]**S[i,j,k,l,m]] -
Eta[a,e] Par[SB[b,c,d]**S[i,j,k,l,m]]
Unprotect[Plus]
x_?NumberQ Epsilon[a_,b_] S_[b_] +
y_?NumberQ Epsilon[a_,c_] S_[c_] :=
(x+y) Epsilon[a,b]**S[b]
x_?NumberQ Epsilon[l_,m_,n_,o_] S_[m_,o_] +
y_?NumberQ Epsilon[l_,m_,n_,p_] S_[m_,p_] :=
(x+y) Epsilon[l,m,n,o]**S[m,o]
x_?NumberQ Epsilon[a_,b_] S_[i_,b_] +

```

```

y_?NumberQ Epsilon[a_,c_] S_[i_,c_] :=
(x+y) Epsilon[a,b] S[i,b]
x_?NumberQ Epsilon[l_,m_,n_,o_] S_[j_,m_,o_] +
y_?NumberQ Epsilon[l_,m_,n_,p_] S_[j_,m_,p_] :=
(x+y) Epsilon[l,m,n,o] S[j,m,o]
x_?NumberQ Epsilon[l_,m_,n_,o_] S_[m_,j_,o_] +
y_?NumberQ Epsilon[l_,m_,n_,p_] S_[m_,j_,p_] :=
(x+y) Epsilon[l,m,n,o] S[m,j,o]
Protect[Plus]
(* EndPackage[] *)

```

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